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On the algebraic decay of disturbances in a stratified linear shear flow

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In this note we confirm the time dependence of the decay of disturbances in a stratified linear shear flow found by Eliassen, Høiland & Riis (1953) and by Booker & Bretherton (1967). The result differs from those of Case (1960*b*) and of Chimonas (1979) which are incorrect.

1. Introduction

If $U(y), R'(y)$ are the velocity and density gradients of a plane parallel shear flow in the x direction and $\psi(x,y,t)$, $T(x,y,t)$ are the perturbation stream function and temperature respectively, then the non-dimensional, linearized equations of motion for an inviscid fluid are

$$\left(\frac{\partial}{\partial t}+U(y)\frac{\partial}{\partial x}\right)\nabla^2\psi - U''(y)\frac{\partial\psi}{\partial x} = -B^2\frac{\partial T}{\partial x}, \tag{1.1}$$

$$\left(\frac{\partial}{\partial t}+U(y)\frac{\partial}{\partial x}\right)T + R'(y)\frac{\partial\psi}{\partial x} = 0. \tag{1.2}$$

Here B^2 is a representative Richardson number and the Oberbeck–Boussinesq approximation has been applied. Booker & Bretherton (1967), in a discussion of the absorption properties of a critical layer resulting from a forcing at a fixed value of y, noted in passing that when $t \gg 1$ and $B^2 > \frac{1}{4}$ equations (1.1), (1.2) have, for the particular case of a linear shear with $U(y) = y$ and $R'(y) = -1$, a solution of the form

$$\psi(x,y,t) \approx t^{-\frac{3}{2}\pm\nu}\Psi_{\pm}(y)\,e^{-ik(x-yt)}, \tag{1.3}$$

where $\nu^2 = \frac{1}{4} - B^2$ and k is real. Just so, and indeed the result is true for all B^2. This formula was previously established by Eliassen *et al.* (1953) – see also Phillips (1966) – but is however in contradiction with the results of Case (1960*b*) who predicted the form

$$\psi(x,y,t) \approx t^{\nu-\frac{1}{2}}\Psi_1(y), \tag{1.4}$$

when $0 < \nu < \frac{1}{2}$, and the recent result of Chimonas (1979) who obtained

$$\psi(x,y,t) \approx t^{2\nu-1}\Psi_2(y)\,e^{-ik(x-yt)}, \tag{1.5}$$

both of which decay more slowly than (1.3).

It is easily verified that (1.5) does not satisfy (1.1) and (1.2) asymptotically, and neither can (1.4) for any smooth function Ψ_1. It is also surprising that Ψ_2 as given by

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Chimonas depends point-wise on the initial density distribution and not on an integrated form. The purpose of this note is to demonstrate that Eliassen *et al.* and Booker and Bretherton were correct and that (1.3) is appropriate. In fact the solution corresponding to (1.3) for nonlinear shears and density profiles has the form

$$\psi(x,y,t) = t^{\lambda(y)}\,e^{-ik[x-U(y)t]}\left\{\Psi_\pm(y)+f_\pm(y)\frac{\log t}{t}+\frac{g_\pm(y)}{t}+O\left(\frac{(\log t)^2}{t^2}\right)\right\}, \tag{1.6}$$

as may easily be verified by substitution into (1.1) and (1.2), where

$$\lambda = -\tfrac{3}{2}\pm\nu, \quad \nu^2 = \tfrac{1}{4}+B^2R'/U'^2, \tag{1.7}$$

and, unless $B = 0$ and $\lambda = -1$,

$$f_\pm(y) = -\frac{i\lambda\lambda'}{kU'}\Psi_\pm(y); \tag{1.8}$$

$g(y)$ and the successive terms in (1.6) can then be found by direct substitution of (1.6) into (1.1) and (1.2). However if $B = 0$ and $\lambda = -1$, $g(y)$ is arbitrary and

$$f_\pm(y) = \frac{iU''}{kU'^2}\Psi_\pm(y). \tag{1.9}$$

In the following we restrict ourselves to the justification of (1.3) as it stands.

When $B = 0$ and $B^2T = 0$, the homogeneous situation, the coefficient of t^{-1} in (1.3) is zero and (1.3) must be replaced by an expression of the form

$$\psi(x,y,t) \approx t^{-2}\Psi(y)\,e^{-ik(x-yt)} \tag{1.10}$$

a result in agreement with that of Orr (1907) and of Engevik (1966) (see also Brown & Stewartson 1978) but not with that of Case (1960*a*). Indeed (1.10) can be deduced at once by inspection of

$$\nabla^2\psi = G(x-yt,y) \tag{1.11}$$

which for arbitrary $G(x,y)$ is the general exact solution of (1.1) when $B = B^2T = 0$, $U = y$.

2. A particular integral for a linear shear

If at $t = 0$, the quantities ψ and T are functions of x alone so that

$$\psi(x,y,0) = \alpha_1(x), \quad T(x,y,0) = \alpha_2'(x) \tag{2.1}$$

then a particular integral of (1.1), (1.2) when $U(y) = y$ and $R'(y) = -1$ may be verified to be

$$T(x,y,t) = f_1(t)\,\alpha_1'(x-yt)+f_2(t)\,\alpha_2'(x-yt), \tag{2.2}$$

$$\psi(x,y,t) = f_1'(t)\,\alpha_1(x-yt)+f_2'(t)\,\alpha_2(x-yt), \tag{2.3}$$

where

$$f_1(t) = tF\left(\frac{3}{4}-\frac{\nu}{2},\frac{3}{4}+\frac{\nu}{2},\frac{3}{2},-t^2\right), \quad f_2(t) = F\left(\frac{1}{4}-\frac{\nu}{2},\frac{1}{4}+\frac{\nu}{2},\frac{1}{2},-t^2\right). \tag{2.4}$$

Here F is the hypergeometric function in the usual notation, $\nu^2 = \frac{1}{4} - B^2$ and ν may be real or imaginary. Now for large t,

$$f_1'(t) \approx \frac{\pi^{\frac{1}{2}}}{2}\left\{\frac{(\nu-1)!\,(\nu-\frac{1}{2})}{\left[\left(\frac{\nu}{2}-\frac{1}{4}\right)!\right]^2}t^{-\frac{3}{2}+\nu} + \frac{(-\nu-1)!\,(-\nu-\frac{1}{2})}{\left[\left(-\frac{\nu}{2}-\frac{1}{4}\right)!\right]^2}t^{-\frac{3}{2}-\nu}\right\}, \tag{2.5}$$

$$f_2'(t) \approx \pi^{\frac{1}{2}}\left\{\frac{(\nu-1)!\,(\nu-\frac{1}{2})}{\left[\left(\frac{\nu}{2}-\frac{3}{4}\right)!\right]^2}t^{-\frac{3}{2}+\nu} + \frac{(-\nu-1)!\,(-\nu-\frac{1}{2})}{\left[\left(-\frac{\nu}{2}-\frac{3}{4}\right)!\right]^2}t^{-\frac{3}{2}-\nu}\right\}, \tag{2.6}$$

from which the asymptotic form of ψ for $t \gg 1$ follows immediately and is seen to have the behaviour predicted in (1.3).

If $B^2 \ll 1$, (2.2), (2.3) reduce to

$$T(x,y,t) = (\tan^{-1}t)\,\alpha_1'(x-yt) + \alpha_2'(x-yt), \tag{2.7}$$

$$\psi(x,y,t) = \frac{1}{1+t^2}\alpha_1(x-yt) - \frac{B^2t}{1+t^2}\alpha_2(x-yt) \tag{2.8}$$

in agreement with (1.10) when $B^2T = O$ and with (1.3) when $B^2T = O(1)$. We note incidentally that the second term in (2.8) eventually dominates if there is any density disturbance no matter how small. We note also that if the initial conditions (2.1) have compact support in x, it is the asymptotic behaviour at fixed $x-yt$ and not at fixed x which is of interest, a point to which Chimonas (1979) correctly drew attention, as did Engevik (1966).

3. The problem for a single Fourier component with y-dependent initial conditions

When the initial values of ψ and T depend on y as well as x then a solution as simple as (2.2), (2.3) does not appear to exist, but we may examine the solution of the initial value problem for one Fourier component which without loss of generality we may take to be e^{-ikx}. Thus with

$$\psi(x,y,t) = e^{-ikx}\phi(y,t), \quad T(x,y,t) = e^{-ikx}S(y,t), \tag{3.1}$$

we have

$$\left(\frac{\partial}{\partial t} - iky\right)\left(\frac{\partial^2\phi}{\partial y^2} - k^2\phi\right) = B^2ikS, \quad \left(\frac{\partial}{\partial t} - iky\right)S + ik\phi = 0, \tag{3.2}$$

together with suitable initial conditions, for example

$$\phi(y,t) = \phi_0(y), \quad S(y,t) = S_0(y) \quad \text{at} \quad t = 0 \tag{3.3}$$

and, for unbounded flow,

$$\phi \to 0 \quad \text{as} \quad |y| \to \infty \quad \text{for all} \quad t \geqslant 0. \tag{3.4}$$

If the Laplace transform with respect to t of $\phi(y,t)$ is denoted by $\bar{\phi}(y,s)$ the system (3.2), (3.3) may be written as

$$(ky+is)^2\left(\frac{\partial^2\bar{\phi}}{\partial y^2} - k^2\bar{\phi}\right) + B^2k^2\bar{\phi} = i(ky+is)\,(\phi_0'' - k^2\phi_0) - B^2ikS_0 \tag{3.5}$$

which may be compared with (3.1) of Chimonas (1979). The boundary condition (3.4) is applicable with ϕ replaced by $\bar{\phi}$. In the following we discuss the solution of (3.5) first in the limit $B \to 0$ and then for general B^2.

The x-independent solutions of (1.1), (1.2) correspond to those of (3.2) with k set equal to zero and may be written down immediately. They are clearly independent of t and do not evolve with time. In any subsequent superposition of solutions of the form (3.1) their role may be essential if the initial vorticity or temperature happens to have a non-zero integral with respect to x. In the following, however, we take $k \neq 0$ though if necessary the limiting process may be made in the appropriate results.

4. The limit $B \to 0$

To facilitate comparison with Chimonas we rewrite (3.5) as

$$\frac{\partial^2 \bar{\Phi}}{\partial Y^2} - \bar{\Phi} + \frac{B^2}{(Y+is)^2}\bar{\Phi} = \frac{F_0(Y)}{(Y+is)^2} + \frac{F_1(Y)}{Y+is}, \tag{4.1}$$

where

$$Y = ky, \quad \bar{\Phi}(Y,s) = \bar{\phi}(y,s), \tag{4.2}$$

$$F_0(Y) = -\frac{iB^2}{k} S_0(y), \quad F_1(Y) = \frac{i}{k^2}\{\phi_0''(y) - k^2\phi_0(y)\}. \tag{4.3}$$

We now set $B = 0$ on the left-hand side and shall find that if it is assumed that $F_0(Y) \neq 0$ then $\phi = O(t^{-1})$ as $t \to \infty$. Under the same assumption Chimonas' result is $\phi = O(1)$. When $B^2 S = 0$ also, so that $F_0(Y) = 0$, we shall find that $\phi = O(t^{-2})$.

With $B = 0$ the solution of (4.1) such that $\bar{\Phi} \to 0$ as $|Y| \to \infty$ is

$$\bar{\Phi}(Y,s) = -\tfrac{1}{2}e^{Y}\int_Y^\infty e^{-Y'}\left\{\frac{F_0(Y')}{(Y'+is)^2} + \frac{F_1(Y')}{Y'+is}\right\}dY'$$
$$-\tfrac{1}{2}e^{-Y}\int_{-\infty}^{Y} e^{Y'}\left\{\frac{F_0(Y')}{(Y'+is)^2} + \frac{F_1(Y')}{Y'+is}\right\}dY'. \tag{4.4}$$

On inverting the Laplace transforms in (4.4) we obtain

$$2\Phi(Y,t) = t\,e^{itY}\int_0^\infty e^{-p}\{F_0(Y+p)\,e^{ipt} + F_0(Y-p)\,e^{-ipt}\}\,dp$$
$$+ i\,e^{itY}\int_0^\infty e^{-p}\{F_1(Y+p)\,e^{ipt} + F_1(Y-p)\,e^{-ipt}\}\,dp, \tag{4.5}$$

where $\Phi(Y,t) = \phi(y,t)$. The result (4.5) is exact for all t and its asymptotic form for large t may be obtained by integrating by parts. On doing this it is found that the terms $O(1)$ in the two integrals involving F_0 cancel as do the terms $O(t^{-1})$ in the two integrals involving F_1. The final result is

$$\Phi(Y,t) = F_0(Y)\frac{e^{itY}}{t}\{1+O(t^{-1})\} + iF_1(Y)\frac{e^{itY}}{t^2}\{1+O(t^{-1})\} \tag{4.6}$$

to which Chimonas' formula (A 12) should reduce as $\nu \to \frac{1}{2}$. The reason that it does not is that the replacement of s by $iy'+\epsilon$ and subsequent treatment of ϵ as a constant in (A 10) is unjustified. A similar error is made by Case (1960*b*).

To compare (4.6) with (2.8) we identify $\phi(y,t)$ with the Fourier transform of $\psi(x,y,t)$ with respect to x and note that (4.3) and (2.1) imply that

$$\bar{\alpha}_1(k) = iF_1(Y), \quad \bar{\alpha}_2(k) = -F_0(Y)/B^2, \tag{4.7}$$

where $\bar{\alpha}_1(k)$, $\bar{\alpha}_2(k)$ are the transforms of $\alpha_1(x)$, $\alpha_2(x)$ respectively. But, from (2.8),

$$\phi(y,t) = \frac{1}{1+t^2}\, e^{iktv}\bar{\alpha}_1(k) - \frac{B^2 t}{1+t^2}\, e^{iktv}\bar{\alpha}_2(k) \tag{4.8}$$

which is exactly the result obtained from (4.6) if F_0, F_1 are replaced by the constant values in (4.7). This example also illustrates how the Fourier representation enables the convective stability of the flow to be examined in co-ordinates in which $x-ty$ is held fixed as $t\to\infty$, a point already made by Engevik (1966).

With F_0 set equal to zero equation (4.6) is analogous to the result obtained by Orr (1907) and Engevik (1966) for a homogeneous fluid bounded in the y direction.

5. The case $B \neq 0$

When $B \neq 0$ an explicit expression as simple as (4.5) for Φ does not exist but two asymptotic approaches are possible which both lead to the same answer and reduce to (4.6) in the limit $B\to 0$. In the first method the differential equation is subjected to an asymptotic analysis and in the second the exact transform of Φ is examined as $t\to\infty$. We do not present the details of the first method here but, for the second, note that the solution of (4.1) such that $\bar{\Phi}\to 0$ as $|Y|\to\infty$ is, when written in a form convenient for analysis as $t\to\infty$,

$$\begin{aligned} B^2\bar{\Phi}(Y,s) &= F_0(Y) + (Y+is)F_1(Y) \\ &+ \frac{1}{\pi}(Y+is)^{\frac{1}{2}}K_\nu(Y+is)\int_{-\infty}^{Y}(-Y'-is)^{\frac{1}{2}}K_\nu(-Y'-is)\,H(Y',s)\,dY' \\ &+ \frac{1}{\pi}(-Y-is)^{\frac{1}{2}}K_\nu(-Y-is)\int_{Y}^{\infty}(Y'+is)^{\frac{1}{2}}K_\nu(Y'+is)\,H(Y',s)\,dY', \end{aligned} \tag{5.1}$$

equivalent to (A 9) of Chimonas' paper. Here

$$H(Y,s) = F_0''(Y) - F_0(Y) + (Y+is)\,[F_1''(Y) - F_1(Y)] + 2F_1'(Y) \tag{5.2}$$

and $F_0(Y), F_1(Y)$ are defined in (4.3). The branch points of (5.1) are at $s = iY$ and in this neighbourhood

$$\begin{aligned} B^2\bar{\Phi}(Y,s) &\approx F_0(Y) \\ &+ \frac{1}{2\sin\nu\pi}\left[\frac{2^\nu(Y+is)^{\frac{1}{2}-\nu}}{(-\nu)!} - \frac{2^{-\nu}(Y+is)^{\frac{1}{2}+\nu}}{\nu!}\right]\int_{-\infty}^{Y}(Y-Y')^{\frac{1}{2}}K_\nu(Y-Y')\,H(Y',iY)\,dY' \\ &- \frac{i}{2\sin\nu\pi}\left[\frac{e^{\nu\pi i}2^\nu(Y+is)^{\frac{1}{2}-\nu}}{(-\nu)!} - \frac{e^{-\nu\pi i}2^{-\nu}(Y+is)^{\frac{1}{2}+\nu}}{\nu!}\right]\int_{Y}^{\infty}(Y'-Y)^{\frac{1}{2}}K_\nu(Y'-Y)\,H(Y',iY)\,dY'. \end{aligned} \tag{5.3}$$

The contributions from the branch points then show that, for $t \gg 1$,

$$\Phi(Y,t) \approx \frac{2^{\nu-1} t^{-\frac{3}{2}+\nu}\, e^{itY}}{B^2(-\nu)!\,(-\frac{3}{2}+\nu)!\sin\nu\pi}\,[e^{(i\pi/2)(\frac{1}{2}-\nu)} I_+ + e^{-(i\pi/2)(\frac{1}{2}-\nu)} I_-]$$
$$- \frac{2^{-\nu-1} t^{-\frac{3}{2}-\nu}\, e^{itY}}{B^2\nu!(-\frac{3}{2}-\nu)!\sin\nu\pi}\,[e^{(i\pi/2)(\frac{1}{2}+\nu)} I_+ + e^{-(i\pi/2)(\frac{1}{2}+\nu)} I_-], \tag{5.4}$$

where

$$I_{\pm}(Y) = \int_0^\infty Y'^{\frac{1}{2}} K_\nu(Y') H(Y \mp Y', iY)\, dY'. \tag{5.5}$$

For the particular case when ψ and T are functions of x alone at $t = 0$, so that (4.7) holds, it may be shown that (5.4) is equal to the asymptotic form for $t \gg 1$ of the Fourier transform of (2.3).

In this note we have not made mention of possible solutions of (1.1), (1.2) proportional to $\exp[-ik(x-ct)]$ for constant c. These would give rise to poles in (5.1) and we suspect that none exist for any $B^2 > 0$ for the problem considered here, though we have not attempted to prove it. When $B = 0$ there are certainly none. They do of course exist for nonlinear profiles which are unstable in the usual sense, and would then dominate (1.6).

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